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LETTER TO THE EDITOR

On the distribution of the effective field for a spin glass on a Cayley tree at zero temperature

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Abstract. An ansatz for the probability distribution of the effective field for a spin glass on a Cayley tree at zero temperature is proposed. The continuous part of the distribution is not differentiable at the origin, in contrast to an ansatz by Katsura and in qualitative agreement with numerical results of de Oliveira. Possible implications are discussed.

The model of a spin glass on a Cayley tree [1–8] is an attractive alternative to the Sherrington–Kirkpatrick model [9]. Indeed, the model's local behaviour is identical to the Bethe approximation, its solution does not involve replicas and it is not plagued by the negative entropy problem of [9]. In addition, and unlike the Sherrington–Kirkpatrick model, it allows for an extensive study of boundary conditions [6–8]. Although the phase diagram has been studied in some detail [7] several problems remain in the low-temperature region.

In this letter, we deal with the distribution of the effective field [2] at zero temperature, a subject of active current interest [8, 10]. The characteristic function (Fourier transform) of this distribution satisfies a nonlinear integral equation (equation (5)) first written down by Katsura [3]. His ansatz for the continuous part of the solution is in the form of a Fourier (spherical) Bessel series [3]. We propose a different ansatz obtained by solving the equation iteratively, starting from Katsura's first approximation. As a consequence, and in contrast to Katsura's ansatz, the solution is a positive linear combination of other characteristic functions, the latter being generated by the process of iteration itself. We now turn to some of the details.

Consider Ising spins on a Cayley tree of coordination number z and suitable boundary conditions (e.g. spins at the boundary equal to +1 [2]). The nearest-neighbour exchange couplings J_{ij} are taken to be independent random variables with common probability distribution $P(J_{ij})=[\delta(J_{ij}+1)+\delta(J_{ij}-1)]/2$. In the limit of an infinite Cayley tree the probability distribution $p(h)$ of the effective field h [2] at a particular internal site is connected to the probability distributions $p(h_k)$ of the effective fields h_k, $k=1,\ldots,z-1$, on that site due to the $(z-1)$ sites of the next generation by the equation

$$p(h)=\int \delta\left[h-f\left(\sum_{k=1}^{z-1} h_k\right)\right]\prod_{k=1}^{z-1} p(h_k)\,\mathrm{d}h_k \qquad (1)$$

with

$$p(h) \geqslant 0 \tag{2a}$$

$$p(h) = p(-h) \tag{2b}$$

$$\int_{-\infty}^{\infty} p(h)\,\mathrm{d}h = 1 \tag{2c}$$

and where, at zero temperature [2-4],

$$f(x) = \mathrm{sgn}(x)\min\{|x|, 1\}. \tag{3}$$

Introducing the characteristic function

$$\phi(x) = \int_{-\infty}^{\infty} \mathrm{d}h\, p(h) \exp(\mathrm{i}\, x\, h) \tag{4}$$

it follows from (1), (3) and (4) that ϕ satisfies the integral equation

$$\phi(x) = (\Lambda\phi)(x) \tag{5}$$

where Λ is the nonlinear integral operator defined by

$$(\Lambda\phi)(x) = \cos x - \frac{\cos x}{\pi}\int_{-\infty}^{\infty} \mathrm{d}y \frac{\sin y}{y}\phi(y)^2 + \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}y \frac{\sin(y-x)}{y-x}\phi(y)^2. \tag{6}$$

This equation should be solved with the constraint that ϕ is a characteristic function, i.e. a function of the form (4) with p satisfying (2) and allowed to be the sum of a linear combination of delta functions at a countable set of points and a continuous function. This is only a special case of the general definition (see, e.g., [11]), where p is just required to be the derivative of an absolutely continuous 'distribution function' which is, in general, only defined almost everywhere and hence is not necessarily continuous [11, section 1.3]. In addition, a Cantor-like 'singular continuous part' [11, section 1.3] could be added (corresponding to p equal to zero almost everywhere). Although our ansatz below (as well as previous ones) only involves this special case, other solutions (in particular, a singular continuous one) cannot be ruled out *a priori.*

The first question which poses itself is whether a solution of (5) exists and is unique: Both questions are answered by exhibiting two solutions:

$$\phi(x) = 1 \tag{7}$$

$$\phi(x) = \tfrac{1}{3} + \tfrac{2}{3}\cos x. \tag{8}$$

Solution (7) corresponds to $p(h) = \delta(h)$ and is therefore the paramagnetic solution; solution (8) is of the spin-glass type [3]. de Oliveira [4] sought solutions to (1) of the form

$$p(h) = \sum_{n=-N}^{N} a_n \delta(h - n/N) \tag{9}$$

with $a_n = a_{-n}$. We obtained the set $\{a_n\}$ numerically for several values of N and thereby found an infinite number of solutions of (1). He also found that the a_n for $n \neq 0$, $n \neq N$, $n \neq -N$ vanish when $N \to \infty$, in such a way that the product Na_n approaches a non-zero constant. Defining $p_c(h) = \lim_{N\to\infty} Na_n$, $n = Nh$, for $n \neq 0$, $n \neq N$, $n \neq -N$, he showed numerically that

$$p(h) = p_d(h) + p_c(h) \tag{10a}$$

where

$$p_d(h) = a\delta(h) + b[\delta(h-1) + \delta(h+1)] \tag{10b}$$

with $a = 0.106\,83$, $b = 0.218\,42$, and p_c continuously differentiable everywhere except at $h = +1$, $h = -1$ and $h = 0$, with

$$\lim_{h\to 0\pm} p_c(h) = \mp 0.0143. \tag{10c}$$

Unfortunately, de Oliveira's solution (9) does not suggest an analytical form of p because the a_n vary with N, as well as the points n/N ($n = -N, \ldots, N$). Katsura [3] proposed an ansatz of the form

$$\phi_K(x) = \phi_{Kd}(x) + \phi_{Kc}(x) \tag{11a}$$

where

$$\phi_{Kd}(x) = a + b \cos x \tag{11b}$$

and

$$\phi_{Kc}(x) = \sum_{l=0}^{\infty} c_{2l} j_{2l}(x) \tag{11c}$$

with the same a and b as de Oliveira's ansatz, and $\{c_{2l}\}_{l=0,1,\ldots}$ coefficients to be determined; the first two are $c_0 = 0.456\,31$ and $c_2 = 0.057\,59$. Above, j_{2l} are spherical Bessel functions and none of them are characteristic functions, except $j_0 = (\sin x)/x$. Indeed, j_{2l}, $l \geq 1$ are all zero at the origin, contradicting (2c) and (4). We therefore set as first approximation (the upper index denoting the order of approximation)

$$\phi_0^{(1)}(x) = \alpha_1 + \beta_1 \cos x + \lambda_1 \frac{\sin x}{x} \tag{12}$$

and attempt to solve (5) iteratively upon requiring that, at each order, the coefficients of those characteristic functions which occur at both the right- and left-hand sides of (5) be equal. Hence, in order to find $\phi_0^{(1)}$ we must compute $\Lambda\phi_0^{(1)}$. By (6), this means that we must calculate $I_x(\phi_0^{(1)})$, where

$$I_x(\phi) \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{d}y \frac{\sin(y-x)}{y-x} \phi(x)^2 \tag{13}$$

for all x, including $x = 0$, which also occurs in the definition (6) of Λ. From (12) and (13), $I_x(\phi)$ (with $\phi = \phi_0^{(1)}$) is a convergent (but not absolutely convergent) Riemann integral which may be computed explicitly as a sum of Cauchy principal values by the residue theorem. In this way, we obtain

$$(\Lambda\phi_0^{(1)})(x) = \tilde{\alpha} + \tilde{\beta} \cos x + \tilde{\lambda} \frac{\sin x}{x} + \mu\phi_{\mathrm{T}}(x) \tag{14}$$

where

$$\phi_{\mathrm{T}}(x) \equiv 2 \frac{(1-\cos x)}{x^2} \tag{15}$$

is the characteristic function of the *triangular* distribution

$$p(h) = \begin{cases} 1 - |h| & -1 \leq h \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

which has, of course, a discontinuous derivative at the origin, and

$$\tilde{\alpha} = \alpha_1^2 + \beta_1^2/2 \tag{16a}$$

$$\tilde{\beta} = 1 - \alpha_1^2 - \beta_1^2/2 - 2\lambda_1\alpha_1 - \lambda_1\beta_1 - 3\lambda_1^2/4 \tag{16b}$$

$$\tilde{\lambda} = 2\lambda_1\alpha_1 + \lambda_1\beta_1 + \lambda_1^2/2 \tag{16c}$$

$$\mu = \lambda_1^2. \tag{16d}$$

The previously defined iterative process requires now in first order that $\tilde{\alpha} = \alpha_1$, $\tilde{\beta} = \beta_1$, and $\tilde{\lambda} = \lambda_1$, whereby (16*a*)-(16*c*) becomes a system of equations for α_1, β_1 and λ_1. Their solution was found numerically to be

$$\alpha_1 = 0.161\,521\,39 \qquad \beta_1 = 0.525\,2014 \qquad \lambda_1 0.288\,7418. \tag{17}$$

Setting $\lambda_1 = 0$ in (16*a*) and (16*b*) we get two equations which, together with the normalisation condition $\alpha_1 + \beta_1 = 1$, yield the two solutions (7) and (8). In order to find $\phi_0^{(2)}$ we must compute $I_x(\Lambda\phi_0^{(1)})$ starting from (13) and (14). The result is

$$\begin{aligned} I_x(\Lambda_0^{(1)}) &= (\tilde{\alpha} + \tilde{\beta}^2/2) + (1 - \tilde{\alpha}^2 - \tilde{\beta}^2/2 - 2\tilde{\lambda}\tilde{\alpha} - \tilde{\lambda}\tilde{\beta} - 3\tilde{\lambda}^2/4)\cos x \\ &\quad + (2\tilde{\lambda}\tilde{\alpha} + \tilde{\lambda}\tilde{\beta} + \tilde{\lambda}^2/2 + \tilde{\beta}\mu + \mu^2/3)\frac{\sin x}{x} \\ &\quad + [\tilde{\lambda}^2/4 + (2\tilde{\alpha} - \tilde{\beta})\mu/2]\phi_{\mathrm{T}}(x) + \tilde{\lambda}\mu\phi_1(x) + \pi\mu^2\phi_2(x) \end{aligned} \tag{18}$$

where

$$\phi_1(x) \equiv \frac{\sin x}{x}\phi_{\mathrm{T}}(x) \tag{19}$$

and

$$\phi_2(x) \equiv \frac{1}{\pi}\left(-\frac{\cos x}{x^2} - \frac{2\sin x}{x^3} + \frac{1}{x^4}(6 - 6\cos x)\right). \tag{20}$$

We see that ϕ_1 is a characteristic function, as a product of two characteristic functions. The Fourier transfrom p_2 of ϕ_2 may be calculated by contour integration:

$$p_2(h) = \begin{cases} 0 & \text{if } h \geq 1 \\ 1 + (|h| - 2)h^2 & \text{if } -1 \leq h \leq 1 \\ 0 & \text{if } h \leq -1. \end{cases} \tag{21}$$

Equation (21) shows explicitly that p_2 satisfies (2) and therefore ϕ_2 is a characteristic function. Further, p_2 has a parabolic behaviour near the origin, which is present in the probability distributions of both [3] and [4]. Putting (18) into the RHS of (6) we find $\Lambda(\Lambda\phi_0^{(1)})$ and, as before, equating the coefficients of those characteristic functions present in *both* $\Lambda(\Lambda\phi_0^{(1)})$ and $\Lambda\phi_0^{(1)}$ we find

$$\phi_0^{(2)} = \alpha_2 + \beta_2\cos x + \lambda_2\frac{\sin x}{x} + \mu_2\phi_{\mathrm{T}}(x) \tag{22}$$

where $\{\alpha_2, \beta_2, \lambda_2, \mu_2\}$ satisfy a simultaneous system of nonlinear algebraic equations which we do not write down, but the solution of which is found to be

$$\begin{aligned} \alpha_2 &= 0.150\,1051 & \beta_2 &= 0.505\,1209 \\ \lambda_2 &= 0.315\,8904 & \mu_2 &= 0.022\,6283. \end{aligned} \tag{23}$$

By (18), $\Lambda\phi_0^{(2)}$ is of type $(\alpha_3 + \beta_3\cos x + \lambda_3(\sin x)/x + \mu_3\phi_{\mathrm{T}}(x) + \nu_3\phi_1(x) + \omega_3\phi_2(x))$ and, although new functions $\phi_3(x), \ldots,$ might (and almost certainly will) apppear in

$\Lambda(\Lambda\phi_0^{(2)})$, again only the coefficients in $\{\alpha_3, \beta_3, \lambda_3, \mu_3\nu_3, \omega_3\}$, common to both $\Lambda\phi_0^{(2)}$ and $\Lambda(\Lambda\phi_0^{(2)})$ will be determined in third order. In this way we obtain a sequence $\phi_0^{(n)}$ of approximations.

We see from the above that each $\phi_0^{(n)}$ is associated with a sequence of positive coefficients $\{\alpha_n, \beta_n, \lambda_n, \ldots\}$ of characteristic functions. We expect that $\phi_0^{(n)}$ converges, as $n \to \infty$, to a characteristic function ϕ_0. Since each $\phi_0^{(n)}$ is a linear combination of characteristic functions with positive coefficients, the limit will be a characteristic function if it is a *convex* combination, i.e. if the sequence of coefficients converges, $\alpha_n \to \alpha$, $\beta_n \to \beta$, $\lambda_n \to \lambda, \ldots$, such that $\alpha + \beta + \lambda + \ldots = 1$ (notice that this normalisation was not required *a priori*). Moreover, the corresponding Fourier transform is manifestly non-negative (property (2*a*)). This is in contrast to ϕ_{Kc} given by (11*c*): the Fourier transforms of the j_l are Legendre polynomials P_l which except for the first $P_0 = 1$ are not everywhere positive; neverthelesss, $\Sigma_{l=0}^{\infty} C_{2l} P_{2l}(x) \geqslant 0$ for $-1 \leqslant x \leqslant l$ because $C_{2l} \ll C_0$ for $l \neq 0$.

In order to verify whether $\{\phi_0^{(n)}\}_{n=1,2,\ldots}$ tends to converge, we compare $\Sigma_{(n)} = \alpha_n + \beta_n + \lambda_n + \mu_n + \ldots$ for successive values of n. By (17), $\Sigma_{(1)} = 0.979\,157$, and, by (23), $\Sigma_{(2)} = 0.993\,7446$, so that it seems that we are going in the right direction. Due to the considerable increase in labour involved to obtain the next approximation, we shall leave a study of $\phi_0^{(n)}$, $n \geqslant 3$, to a further publication. Note, however, that, although (17) and (23) show an evolution towards the coefficients found by de Oliveira (and the first three found by Katsura), comparison with (23) shows that convergence is slow.

We now summarise our conclusions. Because $\mu_2 \neq 0$ in (23), the probability distribution associated with our characteristic function is not differentiable at the origin. In fact, by (10*c*) and (23) the predicted discontinuity in the derivative is of the order of magnitude of the one found in reference [4]. Since Fourier transforms of spherical Bessel functions are Legendre polynomials and the coefficients C_{2l} in (11*c*) seem to decay rapidly with increasing l [3], the probability distribution associated with Katsura's ansatz for the continuous part of the characteristic function does not have a discontinuous derivative at the origin and, therefore, our characteristic function is different from Katsura's. Hence, there exist at least *two* solutions with *non-zero* continuous component. This seems to be of greater interest than the already mentioned existence of an infinity of discrete solutions because, as proved in [10], the solution with continuous component (Katsura's) is longitudinally stable, while the discrete ones are not.

Several open problems remain. We expect that the present solution is also longitudinally stable, which may be verified by the methods of [10]. However, in order to find out which solution has the lowest energy a better control of the behaviour of higher-order coefficients (in both solutions) will be necessary before an unmistakable answer can be provided. In particular, a proof of summability of both sequences of coefficients would render the present (as well as Katsura's) results rigorous.

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References

[1] Thompson C J 1982 *J. Stat. Phys.* **27** 457
[2] Morita T 1979 *Physica* **98A** 566

[3] Katsura S 1986 *Progr. Theor. Phys. Suppl.* **87** 139
[4] de Oliveira M J 1988 *Physica* **148A** 567
[5] Thouless D J 1986 *Phys. Rev. Lett.* **56** 1082
[6] Chayes J T, Chayes L, Sethna J P and Thouless D J 1986 *Commun. Math. Phys,* **106** 41
[7] Carlson J M, Chayes J T, Chayes L, Sethna J P and Thouless D J 1988 *Europhys. Lett.* **55** 355
[8] Kwon C and Thouless D J 1988 *Phys. Rev.* B **37** 7649
[9] Sherrington D and Kirkpatrick S 1975 *Phys. Rev. Lett.* **35** 1792
[10] de Almeida J R L, de Dominicis C and Mottishaw P 1988 *J. Phys. A: Math. Gen.* **21** L693
[11] Chung K L 1968 *A Course in Probability Theory* (New York: Academic)